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ANALOGY BETWEEN EQUATIONS OF PLANE FILTRATION AND EQUATIONS OF LONGITUDINAL SHEAR OF NONLINEARLY ELASTIC AND PLASTIC SOLIDS

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There is a simple analogy between plane problems of nonlinear filtration and problems of longitudinal shear of nonlinearly elastic and plastic solids which makes it possible to transfer results and problem formulations from one field to the other. We formulate this analogy in explicit form (Sect. 1), consider some examples and consequencies (Sect. 2), and justify a variational principle for the equations of nonlinear filtration, which together with the maximum principle yield estimates for the integral characteristics of a filtration stream (Sect. 3).

1. 1°. The system of equations of plane nonlinear filtration of an incompressible fluid consists of the filtration law equations and the continuity equation [1, 2]

$$\operatorname{grad} H = -\Phi(w)\,\mathbf{w}/w, \qquad \operatorname{div}\mathbf{w} = 0 \tag{1.1}$$

where $\mathbf{w}$ is the filtration velocity and H is the pressure head. In the plane problem $\mathbf{w}$ and grad H are two-dimensional vectors lying in the plane x, y.

Now let us consider a cylindrical plastic solid whose generatrix is parallel to the z-axis and assume that every straight line parallel to the generatrix is displaced along itself as a rigid rod. The body is in a state of longitudinal shear (antiplanar straining), the stresses acting in the body are reducible to the two shearing stresses τ_{xz} and τ_{yz}, and (e.g. see [3])

$$\text{grad } \zeta = \boldsymbol{\gamma} = \Gamma(\tau)\, \boldsymbol{\tau} / \tau, \qquad \text{div } \boldsymbol{\tau} = 0 \tag{1.2}$$

Here $\zeta = \zeta(x, y)$ is the displacement along the z-axis and $\boldsymbol{\gamma}$ is the shear strain vector $(\partial\zeta / \partial x, \partial\zeta / \partial y)$, and $\boldsymbol{\tau}$ is the shearing stress vector (τ_{xz}, τ_{yz}). In order to pass from a solid to a nonlinearly viscous fluid we must replace the corresponding displacements by velocities and the strains by straining rates. Equations (1.2) then describe "Couette" motions, i.e. rectilinear parallel motions of the fluid in the absence of a longitudinal pressure gradient.

The functions $\Phi(w)$ and $\Gamma(\tau)$ occurring in Eqs. (1.1) and (1.2) describe the filtration law and the straining or flow law, respectively.

Now let us replace the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ by the vectors $\boldsymbol{\gamma}^*$ and $\boldsymbol{\tau}^*$ which are of the same magnitude but rotated by an angle $\frac{1}{2}\pi$. We have

$$\gamma_x^* = -\gamma_y = -\frac{\partial\zeta}{\partial y}, \quad \gamma_y^* = \gamma_x = \frac{\partial\zeta}{\partial x}, \quad \tau_x^* = -\tau_y, \quad \tau_y^* = \tau_x \tag{1.3}$$

We therefore obtain

$$\text{div } \boldsymbol{\gamma}^* = 0, \qquad \text{rot } \boldsymbol{\tau}^* = 0 \tag{1.4}$$

so that we can consider $\boldsymbol{\tau}^*$ as the gradient of the stress function χ,

$$\boldsymbol{\tau}^* = \text{grad}\chi = S(\gamma^*)\, \boldsymbol{\gamma}^* / \gamma^*, \qquad \text{div } \boldsymbol{\gamma}^* = 0 \tag{1.5}$$

Since the vectors $\boldsymbol{\tau}$, $\boldsymbol{\tau}^*$ and $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}^*$ are of equal magnitude and since the vectors $\boldsymbol{\gamma}^*$ and $\boldsymbol{\tau}^*$ coincide in direction, the function $S(\gamma^*)$ is the inverse of the function $\Gamma(\tau)$.

Comparison of systems (1.1), (1.2) and (1.5) yields two systems of analogies between filtration problems and problems of rectilinear parallel motions. In the first of these analogies the filtration velocity vector corresponds to the stress vector $\boldsymbol{\tau}$, the pressure head gradient taken with the opposite sign to the strain vector $\boldsymbol{\gamma}$, the pressure head H with the opposite sign to the longitudinal displacement ζ, and the stream function ψ to the stress function χ; finally, the function $\Phi(w)$ corresponds to the function $\Gamma(\tau)$,

$$\mathbf{w} \rightleftarrows \boldsymbol{\tau}, \quad -H \rightleftarrows \zeta, \quad -\text{grad } H \rightleftarrows \boldsymbol{\gamma}, \quad \psi \rightleftarrows \chi, \quad \Phi \rightleftarrows \Gamma \tag{1.6}$$

In the second system of analogies the correspondences are

$$\mathbf{w} \rightleftarrows \boldsymbol{\gamma}^*, \quad -H \rightleftarrows \chi, \quad -\text{grad } H \rightleftarrows \boldsymbol{\tau}^*, \quad \psi \rightleftarrows \zeta, \quad \Phi \rightleftarrows S \tag{1.7}$$

2°. Let us establish the correspondences of the singular points and boundary conditions for the solutions of the corresponding problems. In the typical formulation of the problem in filtration theory the boundary C of the domain D in which we seek the solution can consist of segments C_H where the pressure head H assumes a specified value, and segments C_u on which the normal component w_n of the filtration velocity (or, which is the same thing, the derivative $\partial\psi / \partial s$ of the stream function along the boundary) has been specified. Finally, the domain D can contain singular points M_i (sources and sinks) in traversing which the stream function experiences a finite increment q_i and near which the filtration velocity has singularities of the form $q_i / [2\pi\rho(M, M_i)]$, where $\rho(M, M_i)$ is the distance between the point M_i and the present point M.

In the first system of analogies the displacement ζ in the plasticity theory problem is specified on the segment C_H of the boundary, and the stresses τ_n normal to the boundary (or the derivative of the stress function χ along the boundary) are specified on the segment C_w. Finally, the sources and sinks correspond to the concentrated forces q_i, so that the stresses near the points M_i have a singularity of the form $q_i / [2\pi\rho (M, M_i)]$.

In the second system of analogies the values of the stress function χ (and therefore the component of the "additional" stress τ^* along the boundary or the component of the stress τ normal to the boundary) is specified on the segment C_H of the boundary. The values of the displacement (or the component of the strain γ along the boundary) is specified on the segment C_w. In traversing the singular point M_i the displacement ζ increases by the amount q_i. This means that the points M_i must be considered as screw dislocations with the Burgers vector directed along the generatirix and equal to q_i. Finally, the concentrated forces P_i correspond to the point vortices (*) of intensity P_i.

It is clear also that we have a duality of the solutions of nonlinear elasticity (plasticity) problems in pure shear. Specifically, each solution $\tau = \tau(x, y)$, $\gamma = \gamma(x, y)$, $\zeta = \zeta(x, y)$ of system (1.2) corresponds to a solution $\gamma^* = \tau(x, y)$, $\tau^* = \gamma(x, y)$, $\chi = \zeta(x, y)$ provided the function S for the second system coincides with the function Γ for the first.

2. Let us illustrate the above by some examples. **1°.** Sokolovskii [4] pointed out a special case of a nonlinear filtration law for which the mapping of the plane of variables ψ, H onto the plane of the ancillary variables h, θ is conformal. We can express this law in our notation as

$$\Phi(w) = w\,[1 - (w / w_*)^2]^{-1/2} \tag{2.1}$$

Here θ is the angle between the velocity w and the x-axis; h is determined by the relation

$$C\,\frac{e^h}{e^{2h} + C^2 / 4w_*^2} = w \tag{2.2}$$

Neuber [5] obtained a similar result for the problem of longitudinal shear of a nonlinearly elastic solid.

2°. Making use of the hodograph transformation (i.e. taking the stress τ and the angle θ which it forms with the x-axis as the independent variables), Neuber [6] obtained solutions with a singularity for bodies with a wedge-shaped cutout. His equation for the stress function χ is of the form

$$\frac{\tau^2}{\Gamma(\tau)}\frac{\partial}{\partial\tau}\left(\frac{\Gamma^2(\tau)}{\tau\Gamma'(\tau)}\frac{\partial\chi}{\partial\tau}\right) + \frac{\partial^2\chi}{\partial\theta^2} = 0 \tag{2.3}$$

Let us attempt to find the stress distribution over the exterior of the semi-infinite cutout. We can then find a singular solution (i.e. a nontrivial solution for which the stresses at the banks of the cutout are equal to zero) for any form of the function $\Gamma(\tau)$ The required solution is defined to within a constant factor and is of the form

$$\chi = A \sin\theta\,[\Gamma(\tau)]^{-1} \tag{2.4}$$

Making use of the formulas for converting back to the physical plane, we obtain the stress distribution pattern near the cutout.

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*) By a "point vortex" we mean a singularity of the filtration velocity field of the same type as a velocity singularity near a hydrodynamic vortex. Flows with vortices do not arise in filtration theory problems and therefore constitute a theoretical idealization.

Applying the above analogies, we immediately obtain the solutions of two other problems. The substitutions $\tau \to \gamma^*$, $\chi \to \zeta$, $\Gamma \to S$ yield a particular solution of the problem of the displacement distribution near the edge of a rigidly fastened plane in a solid ("a rigid plate welded into a solid").

$$\zeta = A\,[\,S\,(\gamma^*)]^{-1} \sin\theta \tag{2.5}$$

Here θ is the angle between the vector γ^* and the x-axis.

Finally, the substitutions $\tau \to w$, $\chi \to \psi$, $\Gamma \to \Phi$ give us an equation for determining the stream function ψ in the hodograph plane w, θ of the filtration velocity,

$$\frac{w^2}{\Phi(w)}\frac{\partial}{\partial w}\left(\frac{\Phi^2(w)}{w\Phi'(w)}\frac{\partial\psi}{\partial w}\right) + \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{2.6}$$

(which clearly coincides with Eq. (1. 5) of [7]). Neuber's solution gives us the function $\psi(w, \theta)$ for the problem of flow of a filtration stream past a semifinite plate (see [7]),

$$\psi = A\,[\Phi(w)]^{-1} \sin\theta \tag{2.7}$$

The gas dynamic problem of the flow past a plate was solved by Ringleb [8] (see also [9]). The existence of a solution of the form (2. 4) for an arbitrary law $\Gamma(\tau)$ is an immediate consequence of Ringleb's findings.

Finally, Neuber's solutions for wedge-shaped domains can be used directly to describe the behavior of the solution at the corner points of a filtration zone for a power filtration law.

3°. The hodograph transformation can also be applied to problems with a more complex geometry. Papers [10–12] contain formulations of some such problems as well as specific solutions for the case of filtration with a limiting gradient when the function $\Phi(w)$ is of the form

$$\Phi(w) = w + \lambda \quad (w > 0), \qquad 0 \leqslant \Phi(w) \leqslant \lambda \quad (w = 0) \tag{2.8}$$

It is clear that all these problems can be interpreted simply as problems of longitudinal shear of a rigidly plastic body with strengthening for which the stress-strain relationship is of the form

$$S(\gamma) = \gamma + \lambda \quad (\gamma > 0), \qquad 0 \leqslant S(\gamma) \leqslant \lambda \quad (\gamma = 0) \tag{2.9}$$

Here the sources correspond to helical dislocations of the corresponding magnitude, the stagnation zones to rigid cores, and the flux lines to lines of constant displacements ζ. Specifically, the solution obtained in [11] defines the field of a helical dislocation with the Burgers vector q centrally located in a layer whose surfaces are displaced by the distance $q/2$ relative to each other.

Similarly, the particular solution of problems of filtration with a limiting gradient obtained in [7] can be interpreted as solutions of problems on the straining of wedge-shaped solids with a fixed edge and a stress-strain relation of the form (1. 9) and also of problems on the longitudinal flow of viscoplastic (Bingham) fluids in wedge-shaped domains.

When the exact solution of the problem with complex geometry turns out to be unobtainable, it may be expedient to apply the hodograph method and then to solve the problem approximately in the hodograph plane.

For example, let us consider the straining of a layer (Fig. 1) of viscoplastic material with linear strengthening (relation (2. 9)) when a concentrated force of magnitude P per unit length along the z-axis is applied to the middle surface of the layer (perpendicularly to the plane of the Fig. 1) ("Drawing of a rigid filament out of a layer"); the side surfaces of the layer are rigidly fastened. The filtration analog of this problem is that of flow with a limiting gradient produced by a point vortex of intensity P lying midway between two impermeable straight lines. Such flow involves a stagnation zone which corresponds to the rigid domain (the shaded area in Fig. 1) in the plastic material. The

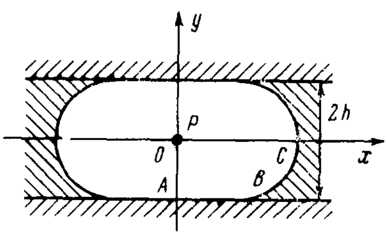

Fig. 1

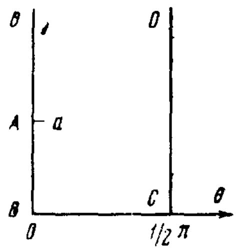

Fig. 2

mapping of the element $OABC$ of the deformed domain into the hodograph plane γ, θ is shown in Fig. 2, and

$$\zeta = 0 \text{ along } ABC, \ \partial\psi / \partial\theta = 0 \text{ along } OA, OC \tag{2.10}$$

In addition, for $\gamma \to \infty$ the solution has a singularity corresponding to a concentrated force, i.e. as we can easily show

$$\zeta = (P / 2\pi) \ln \gamma \tag{2.11}$$

The equation for the displacements ζ for a body conforming to relation (2.9) becomes the equation

$$\gamma(\gamma+\lambda)\frac{\partial^2\zeta}{\partial\gamma^2} + (\gamma-\lambda)\frac{\partial\zeta}{\partial\gamma} + \frac{\partial^2\zeta}{\partial\theta^2} = 0 \tag{2.12}$$

An approximate solution of mixed problem (2.10)–(2.12) which arises in the hodograph plane is easy to obtain if the quantity a, i.e. the strain at the point A of the boundary lying directly under the applied force, is large compared with the characteristic strain λ, $a \gg \lambda$. To obtain the solution in the first approximation we need merely use the analogy with the filtration theory problem and the results of [12], replacing the distribution of ζ over the segment OA (Fig. 2) by the distribution corresponding to the zero value of λ. For $\lambda = 0$ we obtain

$$\zeta = \zeta_0 = \frac{P}{2\pi} \operatorname{Re} \ln \frac{\gamma e^{i\theta} + (\gamma^2 e^{2i\theta} - a^2)^{1/2}}{a} \tag{2.13}$$

Along the line $\gamma = a$ we have

$$\zeta_0 = \frac{P}{2\pi} \operatorname{Re} \ln (e^{i\theta} + (e^{2i\theta} - 1)^{1/2}) \tag{2.14}$$

In the rectangle $0 < \gamma < a$, $0 < \theta < 1/2\,\pi$ the solution of Eq. (2.12) satisfying the boundary conditions at the sides $\theta = 0$, $\theta = 1/2\,\pi$, $\gamma = 0$ is of the form

$$\zeta = \gamma^2 \sum_{m=1}^{\infty} B_m F(2m+1, -2m+3, 3, -\gamma/\lambda) \sin(2m-1)\theta \tag{2.15}$$

(F is a hypergeometric function).

To obtain the first approximation we need merely determine the coefficients B_m from the condition

$$\zeta|_{\gamma=a} = \zeta_0|_{\gamma=a} \tag{2.16}$$

This yields

$$\zeta \approx \sum_{m=1}^{\infty} \left(\frac{\gamma}{a}\right)^2 f_m \frac{F(2m+1, 3-2m, 3, -\gamma/\lambda)}{F(2m+1, 3-2m, 3, -a/\lambda)} \sin(2m-1)\theta \tag{2.17}$$

$$f_m = \frac{4}{\pi} \int\limits_0^{1/2\pi} \zeta_0(a, \theta) \sin(2m-1)\theta \, d\theta \tag{2.18}$$

To determine the boundary of the rigid domain in the first approximation we simply compute the first term of expression (2.17) to obtain

$$f_1 = \frac{2P}{\pi^3} \int\limits_0^{1/2\pi} \sin\theta \operatorname{Re}[\ln(e^{i\theta} + \sqrt{e^{2i\theta} - 1})] \, d\theta = \frac{P}{2\pi} \tag{2.19}$$

Within the error bracket of the approximation the coordinates of the boundary of the rigid domain are given by the expression

$$x + iy = \frac{P}{2\pi} \frac{a+\lambda}{a^2} [\sin^2\theta + i\theta - i\, {}^1/_2 \sin 2\theta] \tag{2.20}$$

The difference $y(^1/_2\pi) - y(0) = h$ and (2.20) gives us

$$h = \frac{P}{4a}\left(1 + \frac{\lambda}{a}\right), \qquad a \approx \frac{P}{4h}\left(1 - \frac{4\lambda h}{P}\right) \tag{2.21}$$

We note that the boundary of the rigid domain tends to some limiting curve as $\lambda / a \to 0$ Its position is completely defined if we also specify the distance of one of its points from the y-axis. We have

$$x_B - x_A = \int\limits_0^a \left(\frac{\partial\psi}{\partial\theta}\right)_{\theta=0} \frac{dw}{w^2}$$

Substituting expression (2.17) into this equation, we obtain

$$x_B - x_A = \frac{\lambda}{a^2} \sum_{m=2}^{\infty} f_m \frac{2m-1}{m(2m-2)} \frac{F(2m,\, 2m-2,\, 2,\, -a_0)}{F(3-2m,\, 1+2m,\, 3,\, -a_0)} + $$
$$+ \lambda a^2 f_1 (1 + a_0) \ln(1 + a_0), \qquad a_0 = a / \lambda \tag{2.22}$$

We can show that the order of the sum of the series as $a_0 \to \infty$ is lower than the order of the isolated term. This allows us to write

$$x_B - x_A \approx \frac{\lambda + a}{a^2} \frac{P}{2\pi} \ln(1 + a_0) = \frac{8}{\pi} h \ln \frac{P}{4\lambda h} \tag{2.23}$$

3. 1°. The analogy between plasticity and filtration problems enables us to carry over the familiar variational principles of the deformational theory of plasticity (nonlinear elasticity) directly into filtration theory. The following statements are valid in the theory of plasticity of incompressible solids [3]. First,

$$A = \int\limits_V T\Gamma dV \tag{3.1}$$

Here A is the work performed by the external forces acting on the body over the statically corresponding displacements; T and Γ are the intensities of the stress and strain tensor deviators. Next, under fixed external loads

$$\delta\left(\int\limits_V \Pi dV - A\right) = 0 \tag{3.2}$$

(the principle of minimum total system energy); here

$$\delta\Pi = T\delta\Gamma \tag{3.3}$$

is the variation of the strain potential Π; the total energy is defined as

$$Э = \int \Pi d \quad - A \tag{3.4}$$

Finally, we have the principle of minimum additional work R under fixed external loads,

$$\delta \int R dV = 0, \qquad R = \int_0^T \Gamma(T)\, dT \tag{3.5}$$

In longitudinal shear

$$\delta \Pi = \tau \delta \gamma, \qquad \delta R = \gamma \delta \tau \tag{3.6}$$

Now let us consider plane filtration motion in a source-free domain D bounded by the contour C. By the second system of analogies (Sect. 1) expression (3.1) gives us

$$\oint_C (\operatorname{grad} H)_s \psi ds = \int_D w \operatorname{grad} H dS \tag{3.7}$$

But

$$\oint_C (\operatorname{grad} H)_s \psi\, ds = -\oint_C H \frac{\partial \psi}{\partial s} ds + [H\psi]_C$$

and since the increment $[H\psi]_C$ associated with traversal of the contour C is equal to zero, it follows that

$$\oint_C (\operatorname{grad} H)_s \psi\, ds = -\oint_C H \frac{\partial \psi}{\partial s} ds = \oint H w_n ds \tag{3.8}$$

and, by (3.7),

$$-\oint_C H w_n ds = -\int_D \operatorname{grad} H \mathbf{w} dS \tag{3.9}$$

The left side of this equation represents the work performed by the external forces on the fluid entering the filtration domain per unit time; the right side is the power dissipation in the filtration domain. Thus, (3.9) is the total dissipation identity: all of the work of the external forces is dissipated in the filtration domain.

Introducing the dissipation potential for the filtration motion

$$D = \int_0^w \Phi(w)\, dw \tag{3.10}$$

by analogy with the plastic potential Π, we obtain (by analogy with (3.2)) the variational equation

$$\delta \left(\int_D D dS + \int_C H w_n ds \right) = 0 \tag{3.11}$$

where the variation is carried out for constant values of the pressure head H along the contour C_H; the permissible velocity fields $\mathbf{w}$ are defined as those fields which satisfy the continuity condition

$$\operatorname{div} \mathbf{w} = 0 \tag{3.12}$$

Specifically, for $(\delta w_n)_C = 0$ we have

$$\delta \left(\int_D D dS \right) = 0 \tag{3.13}$$

i.e. the true velocity field differs from all other velocity fields satisfying the continuity equation and having the same values of the normal component of the velocity at the domain contour in that it minimizes the total dissipation potential,

$$D^* = \int_D D dS \tag{3.14}$$

Finally, by analogy with (3, 5) we find that the true pressure head distribution differs from all the other distributions which assume the same values at the boundary in that it minimizes the total additional dissipation potential,

$$R^* = \int_D R dS, \qquad R = \int_0^Z \Psi(\eta)\, d\eta, \qquad Z = |\operatorname{grad} H| \tag{3.15}$$

where the function Ψ is the inverse of Φ.

2°. A few remarks concerning the above variational principles are in order.

In the first place, these principles can be proved directly without reference to plasticity and not only for plane, but also for three-dimensional filtration motions. It is easy to show that the true motion corresponds to the minimum of the corresponding functional not only with respect to infinitely small distances, but also with respect to all permissible states. This implies the uniqueness of the filtration motion in a finite domain; the values of w_n are specified over part of the boundary of this domain, and the values of H are specified over the other part. All of the proofs can be carried out as in Prager's paper [13]. This involves the use of the familiar Young inequality [14]

$$ab \leqslant \int_0^a f(x)\, dx + \int_0^b \varphi(x)\, dx \tag{3.16}$$

where and φ are mutually reciprocal functions. The proofs remain valid for motions with a limiting gradient involving the formation of stagnation zones.

Finally, variational principle (3, 11) can be used to derive the filtration law equations and boundary conditions in the usual way.

If the medium is inhomogeneous, then the filtration law at each point can be written as

$$\operatorname{grad} H = -\Phi(w, h)\frac{\mathbf{w}}{w}, \qquad \mathbf{w} = -\Psi(|\operatorname{grad} H|, h)\frac{\operatorname{grad} H}{|\operatorname{grad} H|} \tag{3.17}$$

where $h = h(x, y, z)$ is the parameter of relative resistance of the medium $\Phi_h' > 0$, $\Psi_h' < 0$. It is easy to see that all of the above variational principles remain valid for an inhomogeneous medium.

3°. Let us consider the filtration domain $ABCD$ bounded by the two streamlines AB and DC and by the equal pressure head lines AD and BC (Fig. 3).

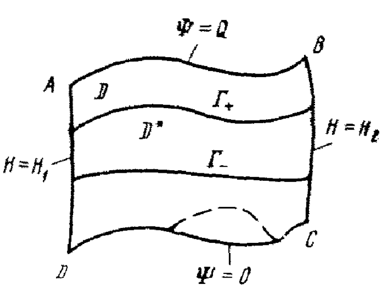

Fig. 3

Let us suppose that the filtration law $\Phi(w, h)$ is of the form

$$\Phi(w, h) = hw^k \tag{3.18}$$

We then infer from (3, 9) that

$$(H_1 - H_2)\, Q = \int_D hw^{k+1} dS \tag{3.19}$$

where Q is the total discharge rate of the filtration stream.

Now let us consider a different field of filtration resistances $h^* \geqslant h$ and the field of velocities w^* associated with it for the previous values of the pressure head at the boundaries AD and BC. Since the field w^* is permissible for the initial domain, we infer from (3.11) that

$$\int_D D(w, h)\, dS - Q(H_1 - H_2) \leqslant \int_D D(w^*, h)\, dS - Q^*(H_1 - H_2) \leqslant$$

$$\leqslant \int_D D(w^*, h^*)\, dS - Q^* (H_1 - H_2) \qquad (3.20)$$

From (3.10) and (3.18) we obtain the expression $D = hw^{k+1} / (k+1)$, so that

$$\int_D D(w, h)\, dS = \frac{1}{k+1} (H_1 - H_2)\, Q \qquad (3.21)$$

We therefore infer from (3.20) that

$$Q^* \leqslant Q \qquad (3.22)$$

This result is physically self-evident; it means that as the filtration resistance in some part of the filtration domain increases, the discharge rate for the same pressure head decreases.

This makes it possible to obtain estimates for the discharge rate of a filtration stream by replacing it by a flow with a simpler geometry. Thus, if the direction of the streamlines is specified arbitrarily (which is equivalent to breaking up the stream into stream tubes by means of impermeable partitions), then the discharge rate for a given pressure distribution at the boundaries of the domain can only diminish. If we are dealing with motion in a layer of constant thickness, then by placing sufficiently closely spaced partitions parallel to the ceiling and floor of the layer we obtain the scheme of a maximally anisotropic layer of zero permeability across the bedding. The motion of a fluid in such a layer proceeds along sublayers without exchange of fluid between the latter. By virtue of what we said above, such flow is not characterized by a flow rate higher than that of the initial flow. Introduction of segments of zero resistance can only increase the discharge rate.

Charnyi (see [15]) made extensive use of such devices in the case of filtration according to the Darcy law ($k = 1$). Specifically, if impermeable boundaries are "impressed" into the filtration zone, then the discharge rate decreases; on the other hand, the "impression" of constant pressure head lines increases the discharge rate. In essence, these statements are close to those used in estimating the limiting loads for rigidly plastic solids (see [3]), which corresponds to the limiting case $k = 0$ in (3.18).

4°. Somewhat weaker statements can be made about an arbitrary filtration law described by an incresing function $\Phi(w)$. It is easy to show that the maximum principle is valid for the pressure head H and for the stream function ψ, so that these functions cannot assume maximum and minimum values inside the domain of motion or on those boundaries where their normal derivatives are equal to zero (a streamline for the pressure head H and a constant pressure head line for the stream function ψ). Now let us consider the domain D' obtainable from the domain D by "impression" of the streamline DC (the broken curve in Fig. 3). Let the discharge rate Q be the same in the two cases. We can now make the following statement:

the inequality

$$\psi' \leqslant \psi \qquad (3.23)$$

is fulfilled at all points of the domain D', so that

$$w' \geqslant w \qquad (3.24)$$

at all points of the boundary AB; the pressure head drop does not decrease at these points,

$$H' \geqslant H \qquad (3.25)$$

finally,

$$w' \leqslant w \qquad (3.26)$$

along the "undeformed" portions of the streamline.

(The author of [16] proved such statements for Darcy filtration and used them to obtain discharge rate estimates for filtration in domains with a complex geometry.) In fact, let $\psi' > \psi$ at some point $M \in D'$.

Then there exists a subdomain $D^* \subset D$, $M \in D^*$ bounded by the line Γ inside which $\psi' > \psi$ ($\psi' = \psi$ at Γ). The line Γ cannot be a closed curve enclosing D^*, since this would mean that $\psi = \psi'$ in D^* by virtue of its uniqueness. For the same reason D^* cannot be contiguous with one equal pressure head line only. The subdomain D^* must therefore be a strip connecting equal pressure head lines (Fig. 3). Along the upper boundary Γ_+ of this half-strip we have

$$\frac{\partial H}{\partial s} = -\Phi(w)\frac{w_s}{w} = -\frac{\Phi(|\operatorname{grad}\psi|)}{|\operatorname{grad}\psi|}\frac{\partial\psi}{\partial n}$$

Moreover, the solutions ψ and ψ' satisfy the following expressions by hypothesis:

$$\frac{\partial\psi'}{\partial s} = \frac{\partial\psi}{\partial s}, \quad \left|\frac{\partial\psi'}{\partial n}\right| < \left|\frac{\partial\psi}{\partial n}\right|$$

Hence,

$$-\frac{\partial H'}{\partial s} + \frac{\partial H}{\partial s} = \frac{\Phi(w')}{w'}\frac{\partial\psi'}{\partial n} - \frac{\Phi(w)}{w}\frac{\partial\psi}{\partial n} = \frac{\Phi(w')}{\Delta'} - \frac{\Phi(w)}{\Delta} \leqslant 0$$

$$\Delta = \left[1 + \left(\frac{\partial\psi/\partial s}{\partial\psi/\partial n}\right)^2\right]^{1/2}$$

This means that

$$(H_1 - H_2)' < H_1 - H_2$$

On the other hand, at the boundary Γ_-

$$\partial\psi'/\partial s = \partial\psi/\partial s, \qquad 0 < \partial\psi/\partial n < \partial\psi'/\partial n$$

so that

$$-\frac{\partial H'}{\partial s} + \frac{\partial H}{\partial s} \geqslant 0, \qquad (H_1 - H_2)' \geqslant H_1 - H_2$$

We see from that that $(H_1 - H_2)' \geqslant H_1 - H_2$ and $\psi' = \psi$ in D^* by virtue of uniqueness. This contradicts our hypothesis, so that $\psi' \leqslant \psi$ everywhere in D' The following relations are therefore fulfilled on AB:

$$w = \left|\frac{\partial\psi}{\partial n}\right| \leqslant w' = \left|\frac{\partial\psi'}{\partial n}\right|, \quad \frac{\partial H}{\partial s} = \Phi(w) \leqslant \frac{\partial H'}{\partial s} = \Phi(w')$$

This implies statements (3.24), (3.25). Statement (3.26) can be proved in similar fashion. Inequality (3.23) also implies that all the streamlines either remain stationary or experience impression towards a stationary streamline. Finally, we note that all of our results remain valid for a strip between two infinitely long streamlines.

It is easy to see that if the "impressed" segment of a streamline lies inside a stagnation zone $\psi = \text{const}$ for the initial flow, then the equality sign applies in all preceding estimates.

A similar argument shows that "impression" of one of the equal pressure head lines into the domain of motion cannot reduce the discharge rate, nor can it reduce the filtration velocity at any point of the unaltered equal pressure head line.

The above statements, which are physically self-evident, enable us to construct estimates of solutions in cases where an exact solution is difficult to obtain.

5°. An important qualitative consequence of our results is the fact that in the case of filtration flows bounded by two streamlines (flow in a finite or infinite strip of possibly varying width) "impression" of one of the streamlines into the domain can only reduce the size of the stagnation zones adjacent to the other streamline. Specifically, if there were no such stagnation zones in the first place, then "impression" cannot produce them.

Example. A stagnation zone arising near a wall during flow from a source of intensity q lying at the distance L from a wall is larger than the stagnation zone produced by the same source lying between two walls (Fig. 4). Since the second problem has an exact solution [11], this gives us the lower estimate for the stagnation zone in the first problem.

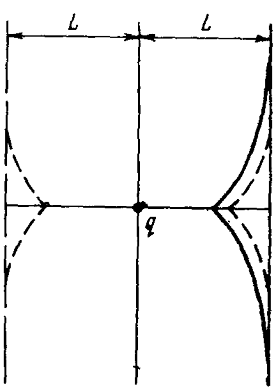

Fig. 4

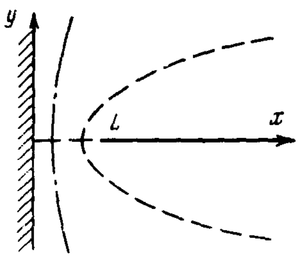

Fig. 5

6°. All of our statements are also valid for the longitudinal shear of plastic solids by virtue of the analogies noted in Sect. 1. In particular, let longitudinal shear be produced by the extraction of a rigid rod of arbitrary cross section from a rigidly plastic solid. Let r be the radius of a circle entirely surrounded by the rod contour and let R be the radius of a circle entirely surrounding the rod cross section. The external rigid zone is then contained in the rigid domain for a circular rod of radius r and contains the rigid domain for a rod of radius R (this is, of course, a self-evident conclusion).

As our next example let us consider the motion produced in a viscoplastic fluid by a half-plane perpendicular to the fixed wall and moving at the velocity U along its rib (Fig. 5). By Sect. 1, the motion in question is an analog of the flow of a filtration stream with the discharge rate U past a wall. It is clear that the flow must tend to radial flow at infinity. This means that the stagnation zones do not arise in the flow (it is quite a simple matter to show that the stagnation zone boundary would have to be concave towards the flow domain, so that the stagnation zone cannot be constructed without violating the conditions at infinity). What we showed in Sect. 5 now implies that stagnation zones cannot arise in flow along the wall of any cylindrical body containing a half-plane $x \geqslant L$.

It is interesting to note that Oldroyd [17] constructed an example of an exact solution in which a stagnation zone extending out to infinity arises during the motion of a body of a certain special shape through Bingham fluid.

The above discussion indicates that a solution without a stagnation zone must exist in addition to Oldroyd's solution. The possibility of two solutions (or, in fact, of an infinite number of solutions) is based on the fact that in formulating a problem for an infinite domain one must specify conditions at infinity. The character of the degeneration of filtration equations with a limiting gradient (and of the analogous equations of longitudinal shear of viscoplastic or rigidly plastic solids) for $w = 0$ is such that boundary conditions must be specified at the line of degeneracy which is, in addition, infinitely far away. This is easy to see by considering problems which admit of mapping onto a hodograph plane. In this lies the fundamental difference between the problems under consideration and problems of linear filtration in which it is enough to require the boundedness of functions at the points of degeneracy.

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